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An alternative formulation of the linearized classical theory of gravitation as a zero mass limit

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1. Introduction and summary

In a preceding paper [1] the present authors have developed a scheme for constructing the matrices of a relativistic wave equation of Dirac form by means of the infinitesimal operators of the 5-dimensional affine group. In this scheme it is not necessary that the mass term is a constant times the unit matrix. It is not necessary, either, that the four coefficient matrices of the momenta form an irreducible set. In fact, the only condition is that these four matrices together with the mass matrix form an irreducible set, and this condition is always possible to satisfy in the frame of the scheme.

It was shown that the scheme gives nothing new in the simplest spin-2 case, labelled by Bhabha [2] as $R_5(2,0)$, and investigated by the authors [3], assuming the most general mass term. However, when the scheme was applied to the spin-2 case, corresponding to Bhabha's case $R_5(2,1)$, it was shown to contain new possibilities beyond Bhabha's theory with general mass term. Especially it was shown that the scheme has in this case a zero mass limit, giving the linearized Einstein equations for the classical (non quantized) gravitational field in empty space and in a general gauge. In this limit the wave function column consists of the components of a trace-less symmetrical tensor of second rank and a scalar, giving together the first approximation of the metric, and of the components of a traceless tensor of third rank and a vector, giving together the superpotential of gravitational theory.

The aim of the present paper is to generalize the case $R_5(2,2)$ of Bhabha by means of the above-mentioned scheme. In this generalized case the wave function column of the Dirac equation consists of the 50 independent components of an affine tensor $\psi^{\alpha\beta\gamma\delta}$, having the same symmetry properties as the Riemann-Christoffel curvature tensor, (2.5), (2.6).

In Section 2 it is shown that the tensor $\psi^{\alpha\beta\gamma\delta}$ is split under R_4 in a tensor, A_{klmn}, (2.7), with the same symmetry properties as the curvature tensor, a tensor, B_{klm}, (2.9), with the same symmetry properties as the superpotential, two symmetric tensors, C_{kl}, (2.13), and D_{kl}, (2.15), a vector, E_k, (2.17), and two scalars, F, (2.18) and G, (2.19), all tensors being trace-less.

In Section 3 the coefficient matrices β_k, (3.4), are constructed by the subspace method of Klein, used in [1,3]. Also, the most general mass term λ is constructed, (3.7).

In Section 4 the Dirac equation is written in component form, (4.1)–(4.7). It is shown, how a special choice of the mass term, (4.8), (4.9), makes the equation reducible into the Bhabha cases $R_5(2,2)$, $R_5(2,0)$ and $R_5(0,0)$. Since the case $R_5(2,2)$ of Bhabha contains no vector or scalar, it is evidently incapable to describe the gravitational case.

In Section 5 the zero mass limit is investigated. In order to obtain the classical gravitational case it proves necessary, as in [1], to generalize the β_k:s of the Dirac equation to the form $\Lambda\beta_k\Lambda$, where Λ, (5.2), is of the same form as the mass term λ, but Hermitian. By putting some of the arbitrary constants of Λ and λ, (5.6), (5.7), equal to zero, the symmetrical tensor D_{kl} and the scalar G disappears, as well as the corresponding components of the wave equation. The remaining Dirac equation has effectively 40 components. By suitably disposing of the remaining arbitrary constants, (5.8), (5.9), this equation is capable of giving the linearized Einstein equations for the gravitational field in a general gauge. For this purpose we have only to connect the metric with the tensor C_{kl} and the scalar F, (5.11), and the superpotential with the tensor B_{klm} and the vector E_k, (5.10). This superpotential is not exactly the same as the superpotential used in [1], but differs by a term, depending on the tensor A_{klmn}, (5.12). This term, however, is shown to give no contribution, (5.20), to the linearized equations for the metric, the term having thus a gauge character. The tensor A_{klmn}, finally, is shown to have a close connection with the Riemann-Christoffel curvature tensor, (5.24).

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2. Properties of the wave function

We consider a wave equation of the form (latin indices running from 1 to 4 and $x^4 = ict$):

$$(\beta^k p_k - \lambda)\psi = 0, \tag{2.1}$$

where

$$p_k = -i\hbar\partial_k = -i\hbar\frac{\partial}{\partial x^k}. \tag{2.2}$$

The quanties β_k and λ are matrices, having the properties developed in Section 2 of [1].

In [1,3] the present authors have investigated in some detail two of the three spin-2 cases, covered by this formalism. In the present paper the authors intend to investigate the remaining spin-2 case.

In this reamining case the wave function consists of the components of a certain 5-dimensional tensor of fourth rank, irreducible under the affine 5-dimensional group. A tensor with the required property is obtained, if a primitive characteristic unit of the symmetric group of permutations of four symbols, constructed from the standard Young tableaux

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{2.3}$$

operates on the indices of an arbitrary tensor $T^{\alpha\beta\gamma\delta}$. One possible construction gives the following irreducible tensor:

$$\begin{aligned}\psi^{\alpha\beta\gamma\delta} = {} & T^{\alpha\beta\gamma\delta} + T^{\gamma\beta\alpha\delta} + T^{\alpha\delta\gamma\beta} + T^{\gamma\delta\alpha\beta} - T^{\beta\alpha\gamma\delta} - T^{\gamma\alpha\beta\delta} - T^{\beta\delta\gamma\alpha} - T^{\gamma\delta\beta\alpha} - \\ & - T^{\alpha\beta\delta\gamma} - T^{\delta\beta\alpha\gamma} - T^{\alpha\gamma\delta\beta} - T^{\delta\gamma\alpha\beta} + T^{\beta\alpha\delta\gamma} + T^{\delta\alpha\beta\gamma} + T^{\beta\gamma\delta\alpha} + T^{\delta\gamma\beta\alpha}. \end{aligned} \tag{2.4}$$

This tensor, which we use as wave function, has the symmetry properties

$$\psi^{\alpha\beta\gamma\delta} = \psi^{\gamma\delta\alpha\beta} = -\psi^{\beta\alpha\gamma\delta} = -\psi^{\alpha\beta\delta\gamma}, \tag{2.5}$$

$$\psi^{\alpha\beta\gamma\delta} + \psi^{\alpha\gamma\delta\beta} + \psi^{\alpha\delta\beta\gamma} = 0, \tag{2.6}$$

i.e. the same properties as the Riemann-Christoffel curvature tensor.

The number of independent index combinations in each antisymmetric index pair of $\psi^{\alpha\beta\gamma\delta}$ is 10. Since the tensor is symmetric in interchanging the two pairs, it has, disregarding the conditions (2.6), the same number of independent index combinations as a symmetric tensor of second rank in 10 dimensions, i.e. 55. Now it is easily seen that (2.6) gives independent extra conditions only for all indices unequal and only one such condition for every set of unequal indices. In five dimensions the extra conditions can consequently be labelled by the missing index number, giving 5 extra conditions. The number of independent components of $\psi^{\alpha\beta\gamma\delta}$ is thus $D_5 = 55 - 5 = 50$.

Under the restriction to R_4 the wave function $\psi^{\alpha\beta\gamma\delta}$ is split in the following seven irreducible tensors:

A) A tensor of fourth rank,

$$\begin{aligned} A_{klmn} = \psi_{klmn} - \tfrac{1}{2}\left(\psi_{rl}{}^{r}{}_{n}\,\delta_{km} + \psi_{rlm}{}^{r}\,\delta_{kn} + \psi_{kr}{}^{r}{}_{n}\,\delta_{lm} + \psi_{krm}{}^{r}\,\delta_{ln}\right) + \\ + \tfrac{1}{6}\,\psi_{rs}{}^{rs}\left(\delta_{km}\,\delta_{ln} - \delta_{kn}\,\delta_{lm}\right), \end{aligned} \tag{2.7}$$

satisfying (2.5), (2.6) and

$$A_{rl}{}^{r}{}_{n} = A_{rlm}{}^{r} = A_{kr}{}^{r}{}_{n} = A_{krm}{}^{r} = A_{r}{}^{r}{}_{mn} = A_{klr}{}^{r} = 0. \tag{2.8}$$

By the same arguments as in 5 dimensions, the tensor A_{klmn} has by virtue of (2.5) the same number of independent index combinations as a symmetric tensor of second rank in 6 dimensions, i.e. 21. The equation (2.6) gives in this case only one extra condition, since there exists only one set of unequal indices. The number of independent extra conditions, arising from (2.8), is the same as the number of independent components of a symmetric tensor of second rank, i.e. 10. The number of independent components of A_{klmn} is thus ${}^{A}d_4 = 21 - 1 - 10 = 10$, corresponding to $R_4(2{,}2)$.

B) A tensor of third rank

$$B_{klm} = \psi_{5klm} - \tfrac{1}{3}\left(\psi_{5r}{}^{r}{}_{m}\,\delta_{kl} + \psi_{5rl}{}^{r}\,\delta_{km}\right), \tag{2.9}$$

satisfying

$$B_{klm} + B_{lmk} + B_{mkl} = 0, \tag{2.10}$$

$$B_{klm} = -B_{kml}, \tag{2.11}$$

$$B_{rl}{}^{r} = B_{r}{}^{r}{}_{m} = B_{kr}{}^{r} = 0. \tag{2.12}$$

This tensor, having the same symmetry properties as the tensor A_{klm} of [1], i.e. the same as the superpotential of gravitational theory, has ${}^{B}d_4 = 16$ independent components, corresponding to $R_4(2,1)$.

C) A symmetric tensor of second rank,

$$C_{kl} = \psi_{5kl5} - \tfrac{1}{4}\, \psi_{5r}{}^{r}{}_{5}\, \delta_{kl}, \tag{2.13}$$

satisfying

$$C_{kl} = C_{lk}, \;\; C_r{}^r = 0, \tag{2.14}$$

having ${}^{C}d_4 = 9$ independent components, corresponding to $R_4(2,0)$.

D) A symmetric tensor of second rank,

$$D_{kl} = \psi_{rk}{}^{r}{}_{l} - \tfrac{1}{4}\, \psi_{rs}{}^{rs}\, \delta_{kl}, \tag{2.15}$$

satisfying

$$D_{kl} = D_{lk}, \;\; D_r{}^r = 0, \tag{2.16}$$

having ${}^{D}d_4 = 9$ independent components, corresponding to $R_4(2,0)$.

E) A vector,

$$E_k = \psi_{5r}{}^{r}{}_{k}, \tag{2.17}$$

having ${}^{E}d_4 = 4$ components, corresponding to $R_4(1,0)$.

F) A scalar,

$$F = \psi_{5s}{}^{s}{}_{5}, \tag{2.18}$$

with ${}^{F}d_4 = 1$, corresponding to $R_4(0,0)$.

G) A scalar,

$$G = \psi_{rs}{}^{rs}, \tag{2.19}$$

with ${}^{G}d_4 = 1$, corresponding to $R_4(0,0)$.

As in [1] we will in the following consider the *A*, *B*, *C*, *D*, *E*, *F*, *G*-components as the independent components of the wave function. The components of $\psi_{\alpha\beta\gamma\delta}$ can be expressed in the *A*, *B*, *C*, *D*, *E*, *F*, *G*-components by the formulae:

$$\psi_{klmn} = A_{klmn} + \tfrac{1}{2}\left(D_{ln}\,\delta_{km} - D_{lm}\,\delta_{kn} + D_{km}\,\delta_{ln} - D_{kn}\,\delta_{lm}\right) + \\ + \tfrac{1}{12}\, G\left(\delta_{km}\,\delta_{ln} - \delta_{kn}\,\delta_{lm}\right), \tag{2.20}$$

$$\psi_{5klm} = B_{klm} - \tfrac{1}{3}\left(E_l\,\delta_{km} - E_m\,\delta_{kl}\right), \tag{2.21}$$

$$\psi_{5kl5} = C_{kl} + \tfrac{1}{4}\, F\, \delta_{kl}, \tag{2.22}$$

$$\psi_{rk}{}^{r}{}_{l} = D_{kl} + \tfrac{1}{4}\, G\, \delta_{kl}. \tag{2.23}$$

3. Representation of the matrices

We now follow the same procedure as in Section 2 of [3] and Section 4 of [1] and identify the transformation

$$\psi' = Q\,\psi = \left(1 + \varepsilon^{\varrho}_{\sigma}\, Q_{\varrho}{}^{\sigma} + \cdots\right)\psi \tag{3.1}$$

with the affine transformation of a tensor of fourth rank:

$$\psi^{\alpha\beta\gamma\delta} + \varepsilon_\sigma^\xi (Q_\xi{}^\sigma \psi)^{\alpha\beta\gamma\delta} = \psi^{\alpha\beta\gamma\delta} + \varepsilon_\sigma^\alpha \psi^{\sigma\beta\gamma\delta} + \varepsilon_\sigma^\beta \psi^{\alpha\sigma\gamma\delta} + \varepsilon_\sigma^\gamma \psi^{\alpha\beta\sigma\delta} + \varepsilon_\sigma^\delta \psi^{\alpha\beta\gamma\sigma}. \tag{3.2}$$

From (3.2) follows:

$$(Q_\xi{}^\sigma \psi)^{\alpha\beta\gamma\delta} = g_\xi{}^\alpha \psi^{\sigma\beta\gamma\delta} + g_\xi{}^\beta \psi^{\alpha\sigma\gamma\delta} + g_\xi{}^\gamma \psi^{\alpha\beta\sigma\delta} + g_\xi{}^\delta \psi^{\alpha\beta\gamma\sigma}. \tag{3.3}$$

Restricting ourselves to R_4, we obtain in the same way as in Section 4 of [1]:

$$\begin{aligned}(\beta_r \psi)_{\alpha\beta\gamma\delta} = i\,(&\delta_{r\alpha} \psi_{5\beta\gamma\delta} + \delta_{r\beta} \psi_{\alpha 5\gamma\delta} + \delta_{r\gamma} \psi_{\alpha\beta 5\delta} + \delta_{r\delta} \psi_{\alpha\beta\gamma 5} - \\ &- \delta_{5\alpha} \psi_{r\beta\gamma\delta} - \delta_{5\beta} \psi_{\alpha r\gamma\delta} - \delta_{5\gamma} \psi_{\alpha\beta r\delta} - \delta_{5\delta} \psi_{\alpha\beta\gamma r}),\end{aligned} \tag{3.4}$$

$$(\nu \psi)_{\alpha\beta\gamma\delta} = \delta_{5\alpha} \psi_{5\beta\gamma\delta} + \delta_{5\beta} \psi_{\alpha 5\gamma\delta} + \delta_{5\gamma} \psi_{\alpha\beta 5\delta} + \delta_{5\delta} \psi_{\alpha\beta\gamma 5}, \tag{3.5}$$

$$\begin{aligned}(q_{rs} \psi)_{\alpha\beta\gamma\delta} = &\delta_{r\alpha} \psi_{s\beta\gamma\delta} + \delta_{r\beta} \psi_{\alpha s\gamma\delta} + \delta_{r\gamma} \psi_{\alpha\beta s\delta} + \delta_{r\delta} \psi_{\alpha\beta\gamma s} - \\ &- \delta_{s\alpha} \psi_{r\beta\gamma\delta} - \delta_{s\beta} \psi_{\alpha r\gamma\delta} - \delta_{s\gamma} \psi_{\alpha\beta r\delta} - \delta_{s\delta} \psi_{\alpha\beta\gamma r}.\end{aligned} \tag{3.6}$$

Now, the remaining arguments and statements of Section 4 of [1] are still valid, only that we have to add a component G, an index δ and a representation $R_4(2,2)$, whereas the representation $R_4(1,1)$ does not occur here.

The most general λ-term is in this case ($N = A, B, C, D, E, F, G$):

$$\lambda = -i\hbar \big(\sum_N \varkappa_N \omega_N + \varkappa_{CD} \omega_{CD} + \overset{*}{\varkappa}_{CD} \omega_{DC} + \varkappa_{FG} \omega_{FG} + \overset{*}{\varkappa}_{FG} \omega_{GF}\big). \tag{3.7}$$

Here the ω_N are projection operators, defined by

$$M\text{-comp. of } \omega_N \psi = \left\{ \begin{array}{l} \text{corresponding } N\text{-component, if } M = N \\ 0 \text{ if } M \neq N \end{array} \right\}. \tag{3.8}$$

Furthermore, the operators ω_{CD} and ω_{DC} are defined by

$$\left. \begin{array}{l} C_{kl}\text{-component of } \omega_{CD} \psi = D_{kl} \\ D_{kl}\text{-component of } \omega_{CD} \psi = 0 \end{array} \right\}, \tag{3.9}$$

$$\left. \begin{array}{l} C_{kl}\text{-component of } \omega_{DC} \psi = 0 \\ D_{kl}\text{-component of } \omega_{DC} \psi = C_{kl} \end{array} \right\}, \tag{3.10}$$

all A, B, E, F, G-components being zero. Similarly, the operators ω_{FG} and ω_{GF} are defined by

$$\left. \begin{array}{l} F\text{-component of } \omega_{FG} \psi = G \\ G\text{-component of } \omega_{FG} \psi = 0 \end{array} \right\}, \tag{3.11}$$

$$\left. \begin{array}{l} F\text{-component of } \omega_{GF} \psi = 0 \\ G\text{-component of } \omega_{GF} \psi = F \end{array} \right\}, \tag{3.12}$$

all A, B, C, D, E-components being zero. We have

$$\overset{+}{\omega}_{CD} = \omega_{DC}, \tag{3.13}$$

$$\overset{+}{\omega}_{FG} = \omega_{GF}. \tag{3.14}$$

4. The wave equation in component form

Inserting (3.4) and (3.7)–(3.12) in the wave equation (2.1), we obtain for the A, B, C, D, E, F, G-components of this equation:

$$\begin{aligned}&\partial_k B_{lmn} - \partial_l B_{kmn} + \partial_m B_{nkl} - \partial_n B_{mkl} -\\ &\quad - \tfrac{1}{2}\partial_r [(B_l{}^r{}_n + B_n{}^r{}_l)\,\delta_{km} - (B_l{}^r{}_m + B_m{}^r{}_l)\,\delta_{kn} - (B_k{}^r{}_n + B_n{}^r{}_k)\,\delta_{lm} + (B_k{}^r{}_m + B_m{}^r{}_k)\,\delta_{ln}] =\\ &\quad = -i\varkappa_A A_{klmn},\end{aligned} \tag{4.1}$$

$$\begin{aligned}&-\partial_r A^r{}_{klm} - (\partial_l C_{km} - \partial_m C_{kl}) + \tfrac{1}{3}(\partial_r C^r{}_m \delta_{kl} - \partial_r C^r{}_l \delta_{km}) - \tfrac{1}{2}(\partial_l D_{km} - \partial_m D_{kl}) +\\ &\quad + \tfrac{1}{6}(\partial_r D^r{}_m \delta_{kl} - \partial_r D^r{}_l \delta_{km}) = -i\varkappa_B B_{klm},\end{aligned} \tag{4.2}$$

$$\partial_r (B_l{}^r{}_k + B_k{}^r{}_l) + \tfrac{1}{3}(\partial_l E_k + \partial_k E_l) - \tfrac{1}{6}\partial_r E^r \delta_{kl} = -i\varkappa_C C_{kl} - i\varkappa_{CD} D_{kl}, \tag{4.3}$$

$$\partial_r (B_l{}^r{}_k + B_k{}^r{}_l) - \tfrac{2}{3}(\partial_l E_k + \partial_k E_l) + \tfrac{1}{3}\partial_r E^r \delta_{kl} = -i\varkappa_D D_{kl} - i\varkappa^*_{CD} C_{kl}, \tag{4.4}$$

$$-\partial_r C^r{}_k + \partial_r D^r{}_k + \tfrac{3}{4}\partial_k F + \tfrac{1}{4}\partial_k G = -i\varkappa_E E_k, \tag{4.5}$$

$$-2\partial_r E^r = -i\varkappa_F F - i\varkappa_{FG} G, \tag{4.6}$$

$$-4\partial_r E^r = -i\varkappa_G G - i\varkappa^*_{FG} F. \tag{4.7}$$

In the same manner as in the cases treated in [1,3], we can derive a higher order differential equation, satisfied by all components of the wave function. Similarly, the differential operator of this equation can be factorized in Klein-Gordon operators, containing the possible mass values. Of course, the ratios between the mass values can now also be arbitrarily varied by varying the mass parameters $\varkappa$.

As proved in Section 2 of [1] the system of equations (4.1)–(4.7) is irreducible, if λ, i.e. the set of $\varkappa$:s, is sufficiently general. If however

$$\varkappa_C = \varkappa_D, \quad \varkappa_{CD} = 0, \tag{4.8}$$

$$\varkappa_F = \varkappa_G, \quad \varkappa_{FG} = 0, \tag{4.9}$$

we obtain from (4.3), (4.4):

$$\partial_r (B_l{}^r{}_k + B_k{}^r{}_l) = -i\varkappa_C \tfrac{1}{3}(2C_{kl} + D_{kl}), \tag{4.10}$$

$$\partial_l E_k + \partial_k E_l - \tfrac{1}{2}\partial_r E^r \delta_{kl} = -i\varkappa_C (C_{kl} - D_{kl}). \tag{4.11}$$

Further, (4.6) and (4.7) give

$$2F - G = 0, \tag{4.12}$$

and for the combination of F and G, occurring in (4.5):

$$-10\,\partial_r E^r = -i\,\varkappa_F (3F + G). \tag{4.13}$$

The system of equations (4.1)–(4.7) is evidently split into three independent systems of equations, (4.1), (4.2), (4.10) and (4.5), (4.11), (4.13) and (4.12), the first containing A_{klmn}, B_{klm}, $2C_{kl} + D_{kl}$ and no vector or scalar, the second containing $C_{kl} - D_{kl}$, E_k, $3F + G$, and the third containing only the scalar $2F - G$. The first system is equivalent to the case, classified by Bhabha [2] as $R_5(2,2)$ and treated by him for the special case of λ being a constant times the unit matrix. The second system is equivalent to the case $R_5(2,0)$ of Bhabha, treated with a general mass-term by the present authors [3]. The third system is the trivial case $R_5(0,0)$.

5. The gravitational zero mass limit

In order to obtain the gravitational zero mass limit, we have to use the same procedure as in Section 6 of [1], i.e. to generalize the Dirac-equation to the form

$$[(\Lambda \beta^k \Lambda)\, p_k - \lambda]\, \psi = 0, \tag{5.1}$$

where ($N = A$, B, C, D, E, F, G):

$$\Lambda = \sum_N k_N \omega_N + k_{CD}\,\omega_{CD} + k^*_{CD}\,\omega_{DC} + k_{FG}\,\omega_{FG} + k^*_{FG}\,\omega_{GF}. \tag{5.2}$$

The reduction, discussed at the end of Section 4, can now be performed in a similar way as in [1]. We have only to make the assumptions (4.8) and (4.9) and pick out the case $R_5(2,2)$ by putting

$$k_E = k_{CD} = 0. \tag{5.3}$$

This makes the left hand side of (4.5) vanish. Moreover, E_k vanish from (4.3) and (4.4). Hence

$$C_{kl} = D_{kl}, \tag{5.4}$$

$$E_k = G = F = 0, \tag{5.5}$$

as required.

In order to obtain the gravitational zero mass limit, we can make the following different choice of the arbitrary constants:

Firstly we put:

$$k_D = k_{CD} = \varkappa_D = \varkappa_{CD} = 0, \tag{5.6}$$

$$k_G = k_{FG} = \varkappa_G = \varkappa_{FG} = 0. \tag{5.7}$$

This means that (4.4) and (4.7) disappear and the symmetrical tensor D_{kl} and the scalar G vanish from the remaining equations (4.1)–(4.7). Consequently only 40 components remain and the matrices of the Dirac equation are now effectively 40×40-rowed.

Secondly we put:

$$\varkappa_B = 2, \quad \varkappa_E = -1, \tag{5.8}$$

$$\varkappa_C = \varkappa_F = 0. \tag{5.9}$$

Introduce moreover the notations

$$V_{klm} = i\,[B_{klm} + \tfrac{1}{3}(E_m\,\delta_{kl} - E_l\,\delta_{km})], \tag{5.10}$$

$$h_{kl} = C_{kl} + \tfrac{1}{4} F\,\delta_{kl}. \tag{5.11}$$

(4.2) and (4.5) now give:

$$V_{klm} = \tfrac{1}{2}(\partial_l h_{mk} - \partial_m h_{lk} + \partial_m h_r{}^r \delta_{kl} - \partial_r h_m{}^r \delta_{kl} - \\ - \partial_l h_r{}^r \delta_{km} + \partial_r h_l{}^r \delta_{km}) + \tfrac{1}{2}\partial_r A^r{}_{klm}. \tag{5.12}$$

Furthermore (4.3) and (4.6) give

$$\partial_r (V_k{}^r{}_l + V_l{}^r{}_k) = 0. \tag{5.13}$$

Since

$$\partial_r \partial_s A^r{}_k{}^s{}_m = \partial_r \partial_s A^r{}_m{}^s{}_k, \tag{5.14}$$

it follows from (5.12) that

$$\partial_r V_k{}^r{}_l = \partial_r V_l{}^r{}_k. \tag{5.15}$$

Consequently (5.13) means that

$$\partial_r V_k{}^r{}_l = 0. \tag{5.16}$$

If we put

$$g_{kl} = \delta_{kl} + \varepsilon\,h_{kl} + \cdots \tag{5.17}$$

and insert the V_{klm} of (5.12) into (5.16), the bracket terms of (5.12) give, just as in [1], all terms of the linearized Einstein equations for the gravitational field in empty space and in the general gauge. However, we obtain in the present case an additional term from the last term of (5.12). Calculating this additional term by means of (4.1), we obtain:

$$\partial_r \partial_s A^r{}_k{}^s{}_m = \frac{i}{2\varkappa_A}[\partial_r \partial^r \partial_s (B_k{}^s{}_m + B_m{}^s{}_k) - \\ - \partial_k \partial_r \partial_s (B^{rs}{}_m + B_m{}^{sr}) - \partial_m \partial_r \partial_s (B^{rs}{}_k + B_k{}^{sr})]. \tag{5.18}$$

By (4.3), (4.6) and (5.6)–(5.9) we have

$$\partial_r (B_l{}^r{}_k + B_k{}^r{}_l) = -\tfrac{1}{3}(\partial_l E_k + \partial_k E_l). \tag{5.19}$$

Inserting (5.19) in (5.18) and making use of (4.6), we obtain:

$$\partial_r \partial_s A^r{}_k{}^s{}_m = 0. \tag{5.20}$$

We are thus, as well as in [1], left with the Einstein linearized equations,

$$\partial_r \partial^r h_{kl} + \partial_k \partial_l h_r{}^r - \partial_k \partial_r h_l{}^r - \partial_l \partial_r h_k{}^r - (\partial_r \partial^r h_s{}^s - \partial_r \partial_s h^{rs})\, \delta_{kl} = 0. \tag{5.21}$$

The equations (4.1) can be considered as equations for the definition of the tensor A_{klmn}, which, of course, is intimately connected with the Riemann-Christoffel curvature tensor,

$$R_{klmn} = \tfrac{1}{2}\,(\partial_l \partial_m h_{kn} - \partial_l \partial_n h_{km} + \partial_k \partial_n h_{lm} - \partial_k \partial_n h_{ln}). \tag{5.22}$$

If we insert B_{klm} from (4.2) into the first 4 terms of (4.1) and from (5.19) in the remaining terms, we obtain by means of (4.5) and (5.5) and by putting

$$\varkappa_A = -2: \tag{5.23}$$

$$R_{klmn} = A_{klmn} + \tfrac{1}{4}\,\partial_r\,[\partial_k A^r{}_{lmn} + \partial_l A_k{}^r{}_{mn} + \partial_m A_{kl}{}^r{}_n + \partial_n A_{klm}{}^r]. \tag{5.24}$$

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